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Fractional integral associated to generalized cookie-cutter set and its physical interpretation

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Abstract. This paper is based on Nigmatullin's study. When the 'residual' memory set is a generalized cookie-cutter set on $[0, T]$, using various hypotheses it is proved that the fractional exponent of a fractional integral is not uniquely determined by the fractal dimension of the generalized cookie-cutter set. It is determined by $\ln P_1/\ln\xi_1$ of self-similar measure (or infinite self-similar measure) μ on this generalized cookie-cutter set, and can run over all positive real numbers.

1. Introduction

When we describe a structure of the evolution of a physical system far from thermodynamic equilibrium: in amorphous materials [2-4], in the description of structural relaxation of high-T_c oxide superconductors [5], in the process of plastic deformation [6] and fracture of solids [7], in the description of solid solutions [8] and the macrostructure of martensite [9], etc, the medium exhibits memory. The existence of memory means that if at time τ a force $f(\tau)$ acts on the system, then there arises a flux J whose magnitude at time $t > \tau$ is given through a memory function $m(\tau)$ by the equation

$$J(t) = \int_0^t m(t-\tau) f(\tau)\,\mathrm{d}\tau. \tag{1}$$

For any given $T \in (0, \infty)$, if the 'residual' memory set is a Cantor's fractal set (or Cantor's k-bars) in $[0, T]$ generated by $\varphi_1 = \xi x, \varphi_2 = \xi x + (1-\xi)T$ (or $\varphi_j = \xi x + (j-1)\xi T + (j-1)\frac{(1-k\xi)T}{k-1}$, $j = 1, 2, \ldots, k$), and if the total number of remaining states in each stage of the division of this set is normalized to unity

$$\begin{aligned} J(t) &\simeq A_\nu T^{-\nu}[\Gamma(\nu)]^{-1}\int_0^t (t-t')^{\nu-1} f(t')\,\mathrm{d}t' \\ &= A_\nu T^{-\nu} t^\nu \hat{D}^{-\nu} f(t) \end{aligned} \tag{2}$$

(obtained by Nigmatullin [1]) where $\nu = \ln 2/\ln(1/\xi)$ (or $\nu = \ln k/\ln(1/\xi)$) is the fractal dimension of Cantor's set (or Cantor's k-bars), $A_\nu = [\sqrt{2}(1-\xi)T]^{-\nu}$ for Cantor's set, and $A_\nu = \exp[-I_1/\ln(1/\xi)]$ where $I_1 = \int_0^\infty \frac{f'(y)}{f(y)} \ln y\,\mathrm{d}y$ and $f(y) = \frac{1-\mathrm{e}^{-ky/(k-1)}}{k(1-\mathrm{e}^{-y/(k-1)})}$ for Cantor's k-bars, $\Gamma(\nu)$ is the gamma function, and where the fractional exponent of fractional integral

$$\hat{D}^{-\nu} f(t) = (\Gamma(\nu))^{-1}\int_0^1 (1-u)^{\nu-1} f(tu)\,\mathrm{d}u \tag{3}$$

$$= [\nu\Gamma(\nu)]^{-1}\int_0^1 f((1-u_1)t)\,\mathrm{d}u_1^\nu \tag{4}$$

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is equal to the fractal dimension of Cantor set $s = \ln 2/\ln\xi$ (or Cantor's k-bar $s = \ln k/\ln\xi$), furthermore, the physical interpretation of the fractional integral is given.

In this paper, we denote the set of all real numbers by $\mathbb{R}$, the set of all complex numbers by $\mathbb{C}$ and the set of all positive integer numbers by $\mathbb{Z}^+$.

If the 'residual' memory set is a self-similar set which is generated by similarities $S_j x = \xi_j x + b_j$ $(0 < \xi_j < 1, b_1 = 0 < b_2 < \cdots < b_K = T(1-\xi_K), j = 1, 2, \ldots, K)$ on $[0, T]$ or a generalized self-similar set which is generated by a family of similarities $\{S_{n,j}(x) = \xi_{n,j}x + b_{n,j} : 0 < \xi_{n,j} < 1, b_{n,j} \in \mathbb{R},\ j = 1, 2, \ldots, K_n\}_{n\in\mathbb{Z}^+}$ on $[0, T]$, in [10], it is proved that the fractional exponent of fractional integral is not uniquely determined by the fractal dimension of the self-similar set or generalized self-similar set, it is determined by $\ln P_1/\ln\xi_1$ of self-similar measure

$$\mu = \sum_{j=1}^{K} P_j\mu\circ S_j^{-1} \qquad 0 < P_j < 1 \qquad \sum_{j=1}^{K} P_j = 1$$

on this self-similar set or of infinite self-similar measure

$$\mu' = \sum_{j=1}^{\infty} P_j\mu'\circ S_j^{-1} \qquad 0 < P_j < 1 \qquad \sum_{j=1}^{\infty} P_j = 1$$

on the generalized self-similar set, and it can run over all positive real numbers. Naturally there exists the problem: Do conclusions in [10] hold when the 'residual' memory set is a generalized cookie-cutter set? In this paper, we obtain a positive answer.

2. Construction of generalized cookie-cutter sets

For any given $T \in (0,\infty)$, denote $E_0 = [0, T]$, $\{\varphi_{n,j}(x) : E_0 \longrightarrow E_0,\ j = 1, 2, \ldots, K_n < \infty\}_{n\in\mathbb{Z}^+}$ is a family of functions satisfying:

(1) $\varphi_{n,j} : E_0 \longrightarrow \varphi_{n,j}(E_0)$ is 1 to 1 mapping and $\mathrm{Int}(\varphi_{n,i}(E_0)) \cap \mathrm{Int}(\varphi_{n,j}(E_0)) = \emptyset (i \neq j)$ for any n and $1 \leqslant i, j \leqslant K_n$.

(2) For all n, the mapping $S_n : \cup_{j=1}^{K_n}\varphi_{n,j}(E_0) \longrightarrow E_0$, defined by $S_n|_{\varphi_{n,j}(E_0)} = \varphi_{n,j}^{-1}$ is $C^{1+\gamma}$ differentiable, i.e. differentiable with a Hölder continuous derivative DS_n satisfying $|DS_n(x) - DS_n(y)| < c_n|x-y|^\gamma$ for some $c_n > 0$, and $|DS_n(x)| > c_0 > 1$ for some constant number c_0 and all $x \in \cup_{j=1}^{K_n}\varphi_{n,j}(E_0)$.

Then $\{S_n\}_{n\in\mathbb{Z}^+}$ is called a sequence of *cookie-cutter maps*. Now $|D\varphi_{n,j}(x)| < c_0^{-1} < 1, \forall x \in E_0$. For a natural number n, let $I_n = \{1, 2, \ldots, K_n\}$, $\Lambda_n = I_1 \times I_2 \times \cdots \times I_n$,

$$E_{j_1 j_2\cdots j_n} = \varphi_{1,j_1}\circ\varphi_{2,j_2}\circ\cdots\circ\varphi_{n,j_n}(E_0) \qquad E(n) = \cup_{j_1 j_2\cdots j_n\in\Lambda_n} E_{j_1 j_2\ldots j_n}.$$

It is obvious that $E_{j_1 j_2\cdots j_n} \subset E_{j_1 j_2\cdots j_{n-1}}$, $E(n) \subset E(n-1)$, then

$$E_T = \cap_{n\geqslant 1}E(n) = \cap_{n\geqslant 1}\cup_{j_1 j_2\cdots j_n\in\Lambda_n} E_{j_1 j_2\cdots j_n} \tag{5}$$

is called the *generalized cookie-cutter set*. From theorems 3 and 4 of [14], we can estimate its fractal dimension s (denote $a_{n,j} = \sup_{x\in E_0}|D\varphi_j(x)|$, $b_{n,j} = \inf_{x\in E_0}|D\varphi_j(x)|$, $\overline{\beta}_n$ and $\underline{\beta}_n$ satisfy equations $\prod_{i=1}^{n}(\sum_{j=1}^{K_i} a_{i,j}^{\overline{\beta}_n}) = 1$ and $\prod_{i=1}^{n}(\sum_{j=1}^{K_i} b_{i,j}^{\underline{\beta}_n}) = 1$, then $\liminf_{n\longrightarrow\infty}\underline{\beta}_n \leqslant s \leqslant \liminf_{n\longrightarrow\infty}\overline{\beta}_n$).

Examples. If $K_n = K$ and $\varphi_{n,j} = \varphi_j$ for any n, then E_T is a cookie-cutter set defined by Bedford [11]. Furthermore, if $K = 2$, $E_1 = [0, \frac{T}{3}]$, $E_2 = [\frac{2T}{3}, T]$ and $S(x) = 3x \bmod T$, then E_T is a Cantor set. Generally, Cantor k-bars and self-similar sets (see [10]) are all cookie-cutter sets, and all cookie-cutter sets are generalized cookie-cutter sets.

3. Memory measures and (infinite) self-similar measures

Now we concretely construct the memory measure on E_T. For a given family of probabilities $\{P_{n,j}\}$ satisfying

$$\sum_{j=1}^{K_n} P_{n,j} = 1 \tag{6}$$

let the memory measure on $E(n)$ be defined by $\mathrm{d}\mu_n(\tau) = m_n(\tau)\,\mathrm{d}\tau$, where

$$m_n(\tau) = \sum_{j_i\ldots j_n\in\Lambda_n} P_{1,j_1}\ldots P_{n,j_n}\chi_{E_{j_1\ldots j_n}}(\tau)/|E_{j_1\ldots j_n}| \tag{7}$$

and

$$\chi_{E_{j_1\cdots j_n}}(\tau) = \begin{cases} 1 & \tau\in E_{j_1\ldots j_n} \\ 0 & \text{otherwise.}\end{cases}$$

Then the support set $\mathrm{supp}(\mu_n)$ of memory measure μ_n is $E(n)$ and $\int_0^T \mathrm{d}\mu_n\,(\tau) = \int_{E(n)} \mathrm{d}\mu_n\,(\tau) = 1$. For any $A\subset E_0$, we also have

$$\int_A \mathrm{d}\mu_n\,(\tau) = \sum_{j=1}^{K_n} P_{n,j}\int_A \mathrm{d}\mu_{n-1}\circ\varphi_{n,j}^{-1}(\tau) \tag{8}$$

and $\mathrm{supp}(\mu_1)\supset\mathrm{supp}(\mu_2)\supset\cdots$.

For any $E_{j_1\cdots j_k}$ and natural number l, we have $\mu_{k+l}(E_{j_1\cdots j_k}) = \mu_k(E_{j_1\ldots j_k})$. For any continuous real function $g(\tau)$ on $\mathbb{R}$, g is uniformly continuous on $\overline{E_0}$, hence for any $\varepsilon>0$, there exists $\delta>0$ such that if $|U'|<\delta$, $U'\subset\mathbb{R}$, we have

$$\max_{\tau\in U'} g(\tau) - \min_{\tau\in U'} g(\tau) < \varepsilon.$$

We want to prove that $\{\int_{\mathbb{R}} g\,\mathrm{d}\mu_k\}_{k=1}^{\infty}$ is a Cauchy sequence. Since $r_{k,j} = \sup_{x\in E_0}|D\varphi_{k,j}(x)| < c_0^{-1} < 1$, $r_k = \max\{r_{k,1},\ldots,r_{k,K_k}\} < c_0^{-1} < 1$, we can take k large enough such that all $|E_{j_1\cdots j_k}|<\delta$. Then

$$\begin{aligned}
\left|\int_{\mathbb{R}} g\,\mathrm{d}\mu_k - \int_{\mathbb{R}} g\,\mathrm{d}\mu_{k+l}\right| &\leqslant \sum_{j_1\ldots j_k\in\Lambda_k}\left|\int_{E_{j_1\ldots j_k}} g\,\mathrm{d}\mu_k - \int_{E_{j_1\ldots j_k}} g\,\mathrm{d}\mu_{k+l}\right| \\
&\leqslant \sum_{j_1\ldots j_k\in\Lambda_k}\left[\left|\int_{E_{j_1\cdots j_k}} g\,\mathrm{d}\mu_k - m_{j_1\ldots j_k}\mu_k(E_{j_1\ldots j_k})\right|\right. \\
&\quad \left.+\left|m_{j_1\cdots j_k}\mu_{k+l}(E_{j_1\cdots j_k}) - \int_{E_{j_1\cdots j_k}} g\,\mathrm{d}\mu_{k+l}\right|\right] \\
&\leqslant \sum_{j_1\cdots j_k\in\Lambda_k} 2(M_{j_1\cdots j_k}\mu_k(E_{j_1\cdots j_k}) - m_{j_1\cdots j_k}\mu_{k+l}(E_{j_1\cdots j_k})) \\
&= \sum_{j_1\cdots j_k\in\Lambda_k} 2(M_{j_1\cdots j_k} - m_{j_1\cdots j_k})\mu_k(E_{j_1\cdots j_k}) \leqslant 2\varepsilon\mu_0(E_0)
\end{aligned}$$

where $M_{j_1\cdots j_k} = \max_{\tau\in E_{j_1\cdots j_k}} g(\tau)$, $m_{j_1\cdots j_k} = \min_{\tau\in E_{j_1\cdots j_k}} g(\tau)$, hence $\{\int_{\mathbb{R}} g\,\mathrm{d}\mu_n\}$ converges as $k\longrightarrow\infty$. It is easy to see that $\lim_{n\longrightarrow\infty}\int_{\mathbb{R}} g\,\mathrm{d}\mu_k$ is a continuous linear functional on the space of continuous functions. From the Rieze representation theorem, there exists a measure μ satisfying $\int_{E_T}\mathrm{d}\mu(\tau) = 1$, $\mathrm{supp}(\mu) = E_T$ such that

$$\lim_{n\longrightarrow\infty}\int_{\mathbb{R}} g\,\mathrm{d}\mu_n = \int_{\mathbb{R}} g\,\mathrm{d}\mu \tag{9}$$

i.e.

$$\mu_n \longrightarrow \mu \text{ (weakly converge).} \tag{10}$$

For any continuous complex function $g_1(\tau)$, we can write $g_1(\tau) = u(\tau) + \mathrm{i}v(\tau)$, where $u(\tau), v(\tau)$ are continuous real functions, hence

$$\lim_{n\longrightarrow\infty} \int_{\mathbb{R}} u \, \mathrm{d}\mu_n = \int_{\mathbb{R}} u \, \mathrm{d}\mu \qquad \lim_{n\longrightarrow\infty} \int_{\mathbb{R}} v \, \mathrm{d}\mu_n = \int_{\mathbb{R}} v \, \mathrm{d}\mu.$$

Hence $\lim_{n\longrightarrow\infty} \int_{\mathbb{R}} g_1 \, \mathrm{d}\mu_n = \int_{\mathbb{R}} g_1 \, \mathrm{d}\mu$. In particular, for $g_1(\tau) = \mathrm{e}^{-p\tau}$, $p \in \mathbb{C}$, we have

$$\lim_{n\longrightarrow\infty} \int_0^\infty \mathrm{e}^{-p\tau} \mathrm{d}\mu_n(\tau) = \int_0^\infty \mathrm{e}^{-p\tau} \, \mathrm{d}\mu(\tau). \tag{11}$$

We call μ the *memory measure* on generalized cookie-cutter set E_T. If $\lim_{n\longrightarrow\infty} K_n = K < \infty$, $\lim_{n\longrightarrow\infty} P_{n,j} = P_j$ and $\lim_{n\longrightarrow\infty} \varphi_{n,j} = \varphi_j$, then from (8), we have

$$\int_A \mathrm{d}\mu(\tau) = \sum_{j=1}^{K} P_j \int_A \mathrm{d}\mu \circ \varphi_j^{-1}(\tau)$$

i.e.

$$\mu(\cdot) = \sum_{j=1}^{K} P_j \mu \circ \varphi_j^{-1}(\cdot). \tag{12}$$

If a probability measure μ on a generalized cookie-cutter set E_T satisfies (12), then μ is called the *self-similar measure* on E_T and $\{P_j\}$ is called the weights. Similar to Hutchinson [12], we can prove the uniqueness of the self-similar measure. Hence by uniqueness, the memory measure μ is the self-similar measure corresponding to weights $\{P_j\}_{j=1}^{K}$.

If $\lim_{n\longrightarrow\infty} K_n = \infty$, $\lim_{n\longrightarrow\infty} P_{n,j} = P_j$ and $\lim_{n\longrightarrow\infty} \varphi_{n,j} = \varphi_j$, then from (8), we have

$$\int_A \mathrm{d}\mu(\tau) = \sum_{j=1}^{\infty} P_j \int_A \mathrm{d}\mu \circ \varphi_j^{-1}(\tau)$$

i.e.

$$\mu(\cdot) = \sum_{j=1}^{\infty} P_j \mu \circ \varphi_j^{-1}(\cdot). \tag{13}$$

4. Flux function and memory function

We consider

$$J_n(t) = \int_0^t m_n(t-\tau) f(\tau) \, \mathrm{d}\tau. \tag{14}$$

If $f(\tau)$ is a generating function, i.e. $f(\tau) = 0$ for $\tau < 0$, $f(\tau)$ has only finite many first class discontinuous points on any $[a, b] \subset [0, \infty)$ and $|f(\tau)| \leqslant M\mathrm{e}^{s_0\tau}$ for $\tau \in [0, \infty)$, $s_0 \geqslant 0$. Performing Laplace transform on both sides of (14), from the product theorem of Laplace transform, we have

$$\mathcal{J}_n(p) = M_n(p) F(p), \; p \in \mathbb{C} \tag{15}$$

where

$$\mathcal{J}_n(p) = \int_0^\infty \exp(-pt) J_n(t)\,\mathrm{d}t \tag{16}$$

$$M_n(p) = \int_0^\infty \exp(-p\tau) m_n(\tau)\,\mathrm{d}\tau. \tag{17}$$

Noting that

$$M_n(p) = \int_0^\infty \mathrm{e}^{-p\tau}\,\mathrm{d}\mu_n(\tau) \tag{18}$$

$$M(p) = \int_0^\infty \mathrm{e}^{-p\tau}\,\mathrm{d}\mu(\tau) \tag{19}$$

from (11), we have

$$\lim_{n\longrightarrow\infty} M_n(p) = M(p). \tag{20}$$

Hence from (15), we have

$$\mathcal{J}(p) = M(p)F(p) \tag{21}$$

where $\mathcal{J}(p) = \lim_{n\longrightarrow\infty} \mathcal{J}_n(p)$.

Now we assume $\varphi_1(x) = \xi_1 x, 0 < \xi < 1$. Denote $E_j = \varphi_j(E_0) = [a_j, b_j]$, $j = 2, 3, \ldots, K$ (or $j = 2, 3, \ldots$), and let $\tilde{\varphi}_j(x) = \varphi_j(x) - a_j$. From the definition of the generalized cookie-cutter set, we know that $0 < a_j < T$, $j = 2, 3, \ldots, K$ (or $j = 2, 3, \ldots$). Hence, from (12) and (13), we have

$$\begin{aligned} M(p) &= P_1 M(\xi_1 p) + \sum_{j=2}^{K} P_j \int \mathrm{e}^{-p\varphi(\tau)}\mathrm{d}\mu(\tau) \\ &= P_1 M(\xi_1 p) + \sum_{j=2}^{K} P_j \mathrm{e}^{-a_j p} \int \mathrm{e}^{-p\tilde{\varphi}_j(\tau)}\,\mathrm{d}\mu(\tau). \end{aligned} \tag{22}$$

and

$$\begin{aligned} M(p) &= P_1 M(\xi_1 p) + \sum_{j=2}^{\infty} P_j \int \mathrm{e}^{-p\varphi(\tau)}\,\mathrm{d}\mu(\tau) \\ &= P_1 M(\xi_1 p) + \sum_{j=2}^{\infty} P_j \mathrm{e}^{-a_j p} \int \mathrm{e}^{-p\tilde{\varphi}_j(\tau)}\,\mathrm{d}\mu(\tau). \end{aligned} \tag{23}$$

Now we want to obtain the approximate solution of function $M(p)$ satisfying (22) or (23). Since $|\int \mathrm{e}^{-p\tilde{\varphi}_j(\tau)}\,\mathrm{d}\mu(\tau)| \leqslant 1$ when $\mathrm{Re}(p) > 0$, hence when $\mathrm{Re}(p)$ is large enough, we have

$$M(p) = P_1 M(\xi_1 p) + \circ(1) \qquad (\text{as } \mathrm{Re}(p) \longrightarrow +\infty). \tag{24}$$

The unique solution of function equation

$$\overline{M}(p) = P_1 \overline{M}(\xi_1 p) \tag{25}$$

has the form

$$\overline{M}(p) = Ap^{-\nu} \tag{26}$$

where A is a constant depending only on $\{P_j\}_{j=1}^K$ and $\{\varphi_j\}_{j=1}^K$ (or $\{P_j\}_{j=1}^\infty$ and $\{\varphi_j\}_{j=1}^\infty$), and

$$\nu = \ln P_1 / \ln \xi_1. \tag{27}$$

Hence when $\mathrm{Re}(p)$ is large enough,

$$M(p) \approx Ap^{-\nu} \tag{28}$$

and

$$\mathcal{J}(p) \approx Ap^{-\nu}F(p). \tag{29}$$

We perform Laplace inversion transform for (28) and (29) and obtain

$$m(\tau) \approx A(\Gamma(\nu))^{-1}\tau^{\nu-1} \tag{30}$$

$$\begin{aligned} J(t) &\approx A(\Gamma(\nu))^{-1}\int_0^t (t-\tau)^{\nu-1} f(\tau)\,\mathrm{d}\tau \\ &= At^{\nu}(\Gamma(\nu))^{-1}\int_0^1 (1-u)^{\nu-1} f(ut)\,\mathrm{d}u \\ &= At^{\nu}D^{-\nu}f \end{aligned} \tag{31}$$

where

$$\begin{aligned} D^{-\nu}f &= (\Gamma(\nu))^{-1}\int_0^1 (1-u)^{\nu-1} f(ut)\,\mathrm{d}u \\ &= (\nu\Gamma(\nu))^{-1}\int_0^1 f((1-u_1)t)\,\mathrm{d}u_1^{\nu} \end{aligned} \tag{32}$$

where (32) is a fractional integral. Then from (31), we establish the connection between the fractional integral and the flux. In particular, if we assume that $\beta \in (0,\infty)$ is the fractal dimension of generalized cookie-cutter set E_T, when $P_1 = \xi_1^{\beta}$ and $\sum_{j=1}^{K} P_j = 1$ (or $P_1 = \xi_1^{\beta}$ and $\sum_{j=1}^{\infty} P_j = 1$), from (27), we have $\nu = \beta$, thus we establish the connection between the fractional integral and the dimension of the generalized cookie-cutter set.

Now we want to ask the question: When can we write $\mathrm{d}\mu(\tau) = m(\tau)\,\mathrm{d}\tau$? In the following we give a sufficient condition. Since μ has compact support set and $M(p)$ is the Laplace transform of μ, then $M(p)$ is an analytic function in $\mathrm{Re}(p) \geqslant 0$. From (24) and (28), when $|p| \longrightarrow \infty$, we obtain that $M(p)$ converges to 0 uniformly with respect to $\arg(p)$. If we assume that $\int_{a-\mathrm{i}\infty}^{a+\mathrm{i}\infty} M(p)\,\mathrm{d}p$ absolutely converges for any $a > 0$, then from Laplace inverse transform theorem, $M(p)$ is the Laplace transform of function

$$m(\tau) = \frac{1}{2\pi\mathrm{i}}\int_{a-\mathrm{i}\infty}^{a+\mathrm{i}\infty} \mathrm{e}^{p\tau} M(p)\,\mathrm{d}p$$

i.e. $M(p) = \int_0^{\infty} m(\tau)\mathrm{e}^{-p\tau}\,\mathrm{d}\tau$, $\mathrm{d}\mu(\tau) = m(\tau)\,\mathrm{d}\tau$. Then from the product theorem of Laplace transform, we have

$$J(t) = \int_0^t m(t-\tau) f(\tau)\,\mathrm{d}\tau.$$

5. Physical interpretations

If the maps $\{\varphi_j\}$ satisfy $\cup_{j=1}^{K}\varphi_j(E_0) = E_0$ (or $\cup_{j=1}^{\infty}\varphi_j(E_0) = E_0$) and $P_1 = \xi_1$, in this case the fractional exponent $\nu = 1$. It follows from (31) that $J(t)$ is related to $f(\tau)$ through the complete integral and corresponds to the case of complete memory. If all $r_j = \sup_{x\in E_0}|D\varphi_j(x)| \longrightarrow 0 (\nu \longrightarrow 0)$, then from (22) (or (23)), it follows that

$$M(p) = P_1 + \sum_{j=2}^{K} P_j \mathrm{e}^{-pa_j} \tag{33}$$

or

$$M(p) = P_1 + \sum_{j=2}^{\infty} P_j \mathrm{e}^{-pa_j}. \tag{34}$$

In the T representation, expression (33) (or (34)) corresponds to ($m(t) = P_1\delta(\tau - 0) + \sum_{j=2}^{K} P_j\delta(\tau - a_j)$ or ($m(t) = P_1\delta(\tau - 0) + \sum_{j=2}^{\infty} P_j\delta(\tau - a_j)$)) a linear combination of K (or infinite) delta functions of P_j intensity localized at the ends of the chosen interval $[0, T]$ and the point $a_2, a_3, \ldots, a_{K-1}$ (or $a_2, a_3, \ldots$), this case corresponds to complete absence of memory. Thus, it also follows from the above analysis that the exponent ν of the fractional integral corresponds to the fraction of preserved states in the process of evolution of the considered physical system and encompasses the cases of completely closed $\nu = 1$) and Markov ($\nu = 0$) system when all states degenerate into finitely many with infinitely high density. An interesting case for analysis is $P_1 = \xi_1^{1/2}$, in this case $\nu = \frac{1}{2}$, which also corresponds to classical diffusion in quasi-one-dimensional semi-infinite systems, in which the connection between the concentration and flux is always expressed through an integral or derivative of only half order [13–15].

From these argument, some physical systems can be described by equations in fractional derivatives must contain channels belonging to some branching fractal structure. This was confirmed in [15], in which an ‘ultraslow’ diffusion equation of the following type was obtained for the main channel:

$$\frac{\partial^\alpha c}{\partial t^\alpha} = \mathcal{D}_x \frac{\partial^\alpha c}{\partial x^\alpha} \qquad 0 < \alpha < 1. \tag{35}$$

The structure of the channels may differ and be generated by definite fractal structure of the medium. In [16–18], such processes were classified as processes with ‘residual’ memory. A process with ‘residual’ memory corresponds to the energy principle formulated by Jonscher [19] for dielectric relaxation in the frequency domain.

From this point of view, transport processes in percolation clusters, fractal trees, and porous systems really must be reanalysed in order to obtain correct transport equations for such systems.

From [20–22], for transport phenomena in random media, $P(r, t)$ the average probability density, that the walker is at distance r at time t from its starting point at time $t = 0$, $\mathrm{i}(r, t)$ the radial probability current, the relation of $\mathrm{i}(r, t)$ and $P(r, t)$ is the following diffusion equation

$$\int_0^t \mathrm{i}(r, \tau)\,\mathrm{d}\tau = r^{d_f - 1} \int_0^t k(t - \tau) P(r, \tau)\,\mathrm{d}\tau \tag{36}$$

where d_f is the fractal dimension of fractal structure. If the set of time t is a generalized cookie-cutter set on $[0, T]$, then from (30), $k(t-\tau) \simeq (t-\tau)^{-\nu}$. Then the integral equation (36) can be written in the following fractional derivative form:

$$\mathrm{i}(r, t) = \text{ constant } \times r^{d_f - 1} \frac{\partial^\nu P(r, t)}{\partial t^\nu} \tag{37}$$

where

$$\frac{\partial^\nu P(r, t)}{\partial t^\nu} = \frac{1}{\Gamma(1-\nu)} \frac{\partial}{\partial t} \int_0^t \frac{P(r, \tau)}{(t-\tau)^\nu}\,\mathrm{d}\tau.$$

Another large class of physical systems in which one can expect the appearance of equations in fractional derivatives is represented by processes with loss due to collisions. We write Newton’s equation in the form

$$\triangle v_i = \frac{1}{m_i} \int_0^t F_i(r, p, \tau)\,\mathrm{d}\tau = \frac{t}{m_i} \int_0^1 F_i(r, p, ut)\,\mathrm{d}u \tag{38}$$

where m_i is the mass of particle i, and F_i is the force of the interaction of particle i with the medium. If the interaction with the medium with generalized cookie-cutter fractal structure is collisional in nature, then the force can be expressed in the form

$$F_i(r, p, \tau) = F_i(r, p, \tau) \sum P_{j_1} \ldots P_{j_n} \chi_{E_{j_1 \cdots j_n}}(\tau)/|E_{j_1 \cdots j_n}|. \tag{39}$$

For a force acting for only a definite of the time, we obtain, repeating the arguments of the previous section,

$$m_i \triangle V_i(t) = A(\Gamma(\nu))^{-1} \int_0^t (t-\tau)^{\nu-1} m_i F_i(r, p, \tau)\, \mathrm{d}\tau. \tag{40}$$

Using the Leibnitz fomula, we can rewrite equation (40) in the more elegant form

$$m_i \frac{\mathrm{d}^\nu(\triangle V_i)}{\mathrm{d}t^\nu} = A F_i \frac{(\nu-1)!}{\Gamma(\nu)}. \tag{41}$$

This equation can be used to describe Brownian motion and loss due to collisions.

Similar to [1], the results of the previous section can also be applied to the Liouville equation.

6. Conclusions

(1) When the generalized cookie-cutter set E_T on $[0, T]$ is given, no matter which self-similar measure (or infinite self-similar measure) the memory measure on E_T is taken, the approximate expression (30) and (31) of memory function and flux function is invariable.

(2) From (27), no matter which self-similar measure (or infinite self-similar measure) the memory measure is taken, the fractional exponent ν of fractional integral is determined only by $\ln P_1/\ln \xi_1$, while it does not depend on the other weight P_j of self-similar measure (or infinite self-similar measure) and other maps $\{\varphi_j\}$.

(3) The fractional exponent ν of fractional integral is equal to the fractal dimension β of generalized cookie-cutter set E_T if and only if $P_1 = \xi_1^\beta$.

(4) When P_1 changes from 0 to 1, ν can run over all positive real numbers.

(5) From section 5, the fractional integral associated to the generalized cookie-cutter set has physical interpretations.

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